# Contents



## 3.5 Theory: IID Normal Random Variables

### 3.5.1 $t$-Distribution

**Student's $t$-Distribution**

**Definition:** $t$-Distribution

Random variable $T \in (-\infty, \infty)$ has a $t$-Distribution with $p$ degrees of freedom if it has the following pdf

$$f_T(t) = \frac{\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{p}{2}\right)} \frac{1}{\sqrt{p\pi}} \frac{1}{(1+t^2/p)^{(p+1)/2}}$$

**Theorem 8.** *If $Z \sim N(0,1)$, a standard normal random variable, and $U \sim \chi^2_n$, a chi-squared random variable with $n$ degrees of freedom, are independent, then*

$$\frac{Z}{\sqrt{\frac{U}{n}}} \sim t_n$$

*Proof.* We provde an outline of the proof only.

- Find distribution of $Y = \sqrt{\frac{U}{n}}$ given $U \sim \chi^2_n$ using change-of-variable.
- Suppose $X = \frac{Z}{Y}$, then seek cdf $F(x) = P(X \leq x)$, recognizing that

$$\{X \leq x\} = \{(z,y) : z/y \leq x\} = \begin{cases} \{(z,y) : z \leq xy\} & \text{when } x > 0 \\ \{(z,y) : z \geq xy\} & \text{when } x < 0 \end{cases}$$

so

$$F_X(x) = \int_{-\infty}^{-} \int_{xy}^{\infty} f(z,y)dzdy + \int_0^{\infty} \int_{-\infty}^{xy} f(z,y)dzdy$$

where

$$f(z,y) = f_Z(z) f_Y(y)$$

and $f_Z(z)$ is the pdf of N$(0,1)$ and $f_Y(y)$ was computed in the first step.

Finally, $f_X(x) = \frac{dF_X(x)}{dx}$.

□

**Properties:**

1. *symmetric.* $f(-t) = f(t)$
2. *Asymptotically* $N(0,1)$. As $n \to \infty$, $t_n \to N(0,1)$. For $n \geq 30$, these two distributions are very similar.
3. *Longer tails.* The $t$-distribution looks like a normal distribution with longer tails.

We now deliver the all-encompassing theorem for iid Normally-distributed samples.

**Theorem 9.** *Suppose $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, then*

1. $\bar{X} \sim N(\mu, \sigma^2/n)$
2. *$\bar{X}$ and $S^2$ are independent.*
3. $T = \frac{\bar{X}-\mu}{S/\sqrt{n}} \sim t_{n-1}$.
4. $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$.

*Proof.* The proof of 1 we already did. It is a consequence of the reproductive property of normal random variables.

The proof of 2 is non-trivial, and we will not cover it.

Proof of 3.

$$\begin{aligned}
\frac{1}{\sigma^2}\sum_{i=1}^{N}(X_i-\mu)^2 &= \sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2 \sim \chi^2_n \\
\frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X}+\bar{X}-\mu)^2 &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(X_i-\bar{X})^2 + \frac{1}{\sigma^2}\sum_{i=1}^{n}(\bar{X}-\mu)^2
\end{aligned}$$

The last equation shows a $\chi^2_n$ random variable on the left is the sum of a random variable with unknown distribution and a $\chi^2_1$ random variable. From the reproductive property of the chi-squared distribution, we conlude the result.

Proof of 4.

Rewrite the statistic as

$$T = \frac{\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}}{\frac{1}{\sqrt{n-1}}\sqrt{\frac{(n-1)S^2}{\sigma^2}}}$$

and recognize

$$\begin{aligned}
U &= \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1) \\
V &= \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}
\end{aligned}$$

so $T$ can be written as

$$T = \frac{U}{\sqrt{\frac{V}{n-1}}}$$

which follows a $t_{n-1}$ distribution by theorem 8. □

### 3.5.2 One Sample $t$ Tests

**Implications**

The all-encompassing theorem yields conclusions with impacts on how we compute confidence intervals and perform hypothesis tests in the absence of population variance information.

**Confidence interval for $\mu$.**

$$\bar{X} \pm t_{n-1}\left(\frac{1+\alpha}{2}\right)\frac{S}{\sqrt{n}}$$

where $t_{n-1}\left(\frac{1+\alpha}{2}\right)$ is the $\left(\frac{1+\alpha}{2}\right)$th quantile of a $t_{n-1}$ distribution.

Example:

Returning to the queue example with $n = 50$, $\bar{X} = 21.5$, and $S = 15$, find the 90% CI for the long-run queue length. Previously, we computed $(18.02, 24.98)$ based on $\phi_{0.95} = 1.64$, a standard normal quantile, but we assumed

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} \sim N(0, 1)$$

Now, we obtain $t_{n-1}\left(\frac{1+\alpha}{2}\right)$ using `R` as `qt((0.9+1)/2, df=40)` and compute CI

$$\bar{X} \pm 1.68 \frac{15}{\sqrt{50}}$$

to produce interval $(17.94, 25.06)$.

**Hypothesis Testing $H_0 : \mu = A$.** Previously, we assumed, erroneously (but approximately), that

$$\frac{\bar{X} - A}{S/\sqrt{n}} \sim N(0, 1)$$

Now, we recognize

$$T = \frac{\bar{X} - A}{S/\sqrt{n}} \sim t_{n-1}$$

so the critical region for rejection becomes $|T| > t_{n-1}\left(\frac{1+\alpha}{2}\right) S/\sqrt{n}$ and the $p$-value $P(|T| > T_{n-1})$, where $T$ is seen as the random variable from $t_{n-1}$ and $T$ is a constant, the observed statistic.

Example:

The sloppy coders example yielded $n = 40$, $\bar{X} = 2.09$ and $S = 0.31$, and we wanted to test $H_0 : \mu = 2$ against $H_A : \mu > 2$. Our previous $p$-value was 0.02. Now, our statistic is

$$\frac{\bar{X} - 2}{0.31/\sqrt{50}}$$

is compared to a $t_{49}$ distribution and the $p$-value becomes 0.023, from `R> 1-pt((2.09-2)/0.31*sqrt(50), df=49).`

**Power Calculations**

The probability that we reject $H_0$ when $H_0$ is not true is the power, denoted by $1 - \beta$. Here is the procedure for power calculations

1. **Identify a specific alternative hypothesis.** A power calculation cannot proceed without specifying an alternative that we speculate may be true if $H_0$ is not. We'll call it $H_a$ (different from the generic alternative $H_A$ concluded if $H_0$ is not true).

2. **Compute critical value for reject $H_0$.** Choose $\alpha$, then assuming $H_0$, determine the sampling distribution for our test statistic $T$ and find the critical value $c$ such that $P(|T| > c) = \alpha$. If the sampling distribution is not symmetric around 0, then find two critical values, one for each tail, placing $\alpha/2$ area under each tail.

3. **Assuming $H_a$, compute $P(|T| > c \mid H_a)$.** This is the hardest step, where we must derive the sampling distribution for our statistic $T$ under the $H_a$ assumption.

Example:

**Power calculations for $Z$ test.** We consider the slightly non-traditional case of power calculations for one-sided tests. Step 1 considers alternative hypotheses restricted to the *side* of interest, and steps 2 and 3 are modified to only consider one tail for one-sided tests.

We focus on the sloppy coders example, where $S = 0.31, n = 50$. Suppose $H_0 : \mu = 2$ with alternative $H_A : \mu > 2$. Find the power to detect $\mu > 2.1$ if $\alpha = 0.05$.

1. Of all the choices of $\mu$ that we hope to detect, the hardest to detect is $\mu = 2.1$, so the specific alternative is $H_a : \mu = 2.1$.
2. With $H_0 : \mu = 2$ in mind, the test statistic is (approximately, because we're using $Z$, though we are estimating $\sigma$)
$$Z = \frac{\bar{X} - 2}{0.31/\sqrt{50}}$$
with rejection when $Z > 1.644854$ (one-sided with $\alpha = 0.05$, given by `R> qnorm(0.95)`). On the original scale, we reject $H_0$ when $\bar{X} > 1.644854 * 0.31/\sqrt{50} + 2 \approx 2.07$.
3. If $H_a$ is true, then $\bar{X} \sim N\left(2.1, 0.31^2/50\right)$. The power is $P(\bar{X} > 2.07 \mid H_a)$, which is given by `R> 1-pnorm(qnorm(0.95)*0.31/sqrt(50)+2, mean=2.1, sd=0.31/sqrt(50))`, which evaluates to $0.7376561$. (You may notice this calculation is a little sensitive if you round too early. `R> 1-pnorm(2.07, mean=2.1, sd=0.31/sqrt(50))` evaluates to $0.7531061$.)

Note, we work on the original scale for ease. Instead, we could recognize that $T \sim N(0.1/0.31 * \sqrt{50}, 1)$ under $H_a$, and the power is given by `R> 1-pnorm(1.644854, mean=0.1/0.31*sqrt(50), sd=1)`, again $0.737656$.