Probability Review Exam

Name: ______________________________

This exam is worth 5% of your course grade.

Rules

- **Do** show your work. You need **not** simplify answers.
- **Do** use prepared "cheatsheets" with definitions, formulae, but *no worked problems, working `R` code, or proofs*. **Don't** use other books, notes, or pieces of paper.

Procedure

1. Do **not** turn this page until directed. Read it thoroughly.
2. Turn in the written portion of your exam, *the cheatsheet(s)*, and any *scrap paper*.

Approximate Quantiles of the Standard Normal Distribution

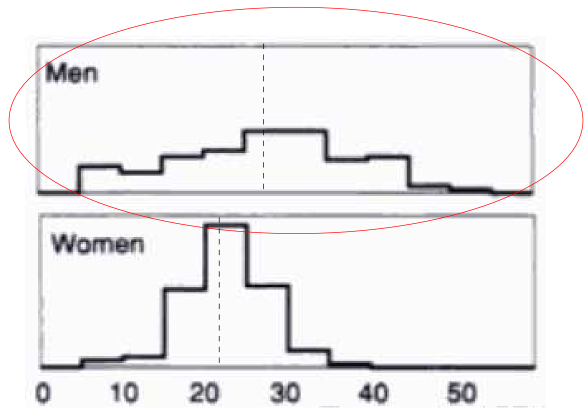
Fraction q	**Quantile** ϕ_q
0.025	-1.96
0.05	-1.64
0.1	-1.28
0.2	-0.84
0.3	-0.52
0.4	-0.25
0.5	0

GOOD LUCK!!

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
5	5	
6	5	
7	5	
8	5	
9	5	
10	5	
11	5	
12	5	
13	5	
14	5	
15	5	
16	5	
Total:	80	

[5 pts] 1. The plot below shows the probability mass functions (pmfs) for the wages earned by men and women at a particular large company. Add a vertical line to each plot indicating where you guess the expectation of each pmf lies. Circle the plot with the larger variance.

Solution:



[5 pts] 2. Name two of the three axioms of probability.

Solution:

1. $P(E) \geq 0$ for any event E.
2. $P(S) = 1$ for the sample space S.
3. $E_1, E_2, \ldots$ are mutually exclusive events, then

$$P(E_1 \cup E_2 \cup \cdots) = \sum_{i=1}^{\infty} P(E_i)$$

[5 pts] 3. You are trying to choose a new color for your apartment walls. If you have to choose from 9 colors and you ask 4 friends for their recommendation, what is the chance that 2 or more people will agree on a color if they have random color preferences?

Solution: We need to find the probability that all 9 friends choose different colors, which is

$$\frac{P_4^9}{9^4} \approx 0.461$$

then the probability that 2 or more will agree on a color is $1 - 0.461 = 0.539$.

[5 pts] 4. The probability you get regular influenza in a season is 0.01. The probability you get H1N1 is 0.10 in a season. The probability you get both in one season is 0.009 (poor you). What is the probability you get sick with flu during the season?

> **Solution:** Use addition rule
>
> $$P(A \cup B) = 0.01 + 0.10 - 0.009 = 0.101$$

[5 pts] 5. Suppose lane 1 takes 80% of cards coming eastbound through an intersection and all other eastbound cars use lane 2. If the probability of getting in an accident in the intersection is 0.001 for lane 1 and 0.01 for lane 2, then what is the probability of an accident occurring to an eastbound car while passing through the intersection.

> **Solution:** Use the law of total probability.
>
> $$P(A \mid L1)P(L1) + P(A \mid L2)P(L2) = 0.001 \times 0.8 + 0.01 \times 0.2 = 0.0028$$
>
> where events are labeled A for accident and Li for lane $i \in \{1, 2\}$.

[5 pts] 6. If Y has pmf $p(1) = \frac{1}{4}, p(2) = \frac{1}{2}$, and $p(3) = \frac{1}{4}$ and

$$g(y) = \begin{cases} 1, & y \geq 3 \\ 0, & \text{otherwise} \end{cases}$$

then what is $E[g(Y)]$?

> **Solution:** $E[g(Y)] = 0 \times \frac{1}{4} + 0 \times \frac{1}{2} + 1 \times \frac{1}{4} = \frac{1}{4}$

[5 pts] 7. The weight W of a truck has pmf $f(w) = \frac{1}{26}$ for $w = \{15, 16, \ldots, 40\}$ tons. The number of trucks Y crossing a bridge has Poisson distribution with mean 50 per hour, independent of their weights. What is the total expected weight passing over the bridge in 1 hour?

> **Solution:** If the total weight passing over the bridge in one hour is T, then $T = W_1 + \cdots W_Y$ and
>
> $$\begin{aligned} E[T] &= E[E[T \mid Y]] \\ &= E[E[W_1 \mid Y] + \cdots + E[W_Y \mid Y]] \\ &= E[E[W \mid Y]Y] = E[E[W]Y] = E[W]E[Y] \\ &= \frac{15 + 16 + \cdots + 40}{26} 50 = 1375 \text{ tons per hour} \end{aligned}$$

The solution given in the 101 questions ($E[T] = E[WY] = E[W]E[Y]$) is not quite correct, but obviously accepted.

[5 pts] 8. Name one assumption of a Binomial random experiment.

Solution:

1. There are n trials, where n is fixed.
2. Each trial results in a success or failure.
3. The probability of success is a constant p.
4. The trials are independent.

[5 pts] 9. Identify one pair of distributions that are approximately equal and the conditions when they are equal.

Solution: We have several choices. When n is large and p is not too small or too large, then

$$\text{Binomial}(n,p) \approx \text{Normal}(np, np(1-p))$$

When N is large (and $n << N$), then

$$\text{Hypergeometric}(N,n,r) \approx \text{Binomial}\left(n, \frac{r}{N}\right)$$

When n is large and p is small, then

$$\text{Binomial}(n,p) \approx \text{Poisson}(np)$$

[5 pts] 10. What would be an appropriate distribution to model the number of accidents at a intersection in a month if it is known that 1 in $10,000$ cycles of the light results in an accident and there is one cycle of the light every 5 minutes?

Solution: The Binomial($N, 1/10000$), where N is the number of light cycles in a month, depending on the month about $30 \times 24 \times 60/5$. This could be approximated with a Poisson($N/1000$).

[5 pts] 11. What is a good distribution to model the number of maple trees sprouting from a 1000 square foot patch if seeds are dropped randomly by birds flying overhead?

> **Solution:** Poisson(λ), where λ is the expected number of birds dropping seeds that sprout during the time of the experiment. If we know the rate (per time and space) at which birds drop seeds that sprout, then we can compute λ. Since the number of birds flying over the patch is random, the Binomial distribution does not work because the number of trials is not known before the random experiment takes place.

[5 pts] 12. If $X \sim \text{Uniform}(10, 40)$, what is $P(X > 30 \mid X > 20)$?

> **Solution:** $\frac{P(X>30)}{P(X>20)} = \frac{1/3}{2/3} = \frac{1}{2}$

[5 pts] 13. If $Y \sim \text{N}(80, 9)$ is the amount of time it take a student to take an exam, how much time should I schedule to insure 97.5% of the students have time to complete the exam?

> **Solution:** The upper 0.025 quantile of a standard normal is 1.96 (because normal is symmetric), which we'll approximate by 2.
>
> $$\frac{Y - 80}{3} = 2 \quad \Rightarrow \quad Y = 86$$

[5 pts] 14. The random variable $Y \sim \text{Gamma}(n, \beta)$ can be thought of as $Y = X_1 + \cdots + X_n$, where X_i are independent exponential random variables with mean β. Use this to prove that $V(Y) = n\beta^2$.

> **Solution:** The variance for sums of random variables gives us
>
> $$V(Y) = V(X_1 + \cdots + X_n) = \sum_{i=1}^{n} V(X_i) + 2 \sum_{1 \le i < j \le n} \text{Cov}(X_i, X_j) = \sum_{i=1}^{n} \beta^2 = n\beta^2$$
>
> where the covariance terms are 0 because of independence and $V(X_i) = \beta^2$ is the variance of an Exponential(β).

[5 pts] 15. Write down (no need to solve) the integral needed for computing $E[Y]$ when (X, Y) is uniform over the triangle with corners $(-1, 0), (0, 1), (1, 0)$.

Solution: The area of the triangle is 1, so $f(x, y) = 1$.

$$\int_{-1}^{0}\int_{0}^{x+1} y dx dy + \int_{0}^{1}\int_{0}^{1-x} y dx dy$$

[5 pts] 16. Are X and Y independent if the joint pdf is $f(x, y) = ax^2\sqrt{x+y}$ for $1 \leq x, y \leq 3$?

Solution: No the joint pdf cannot be factored into a function involving only x and one involving only y, so X and Y cannot be independent.