Experimental Design - Homework Problems

1 To Block or Not to Block?

A statistics instructor at Iowa State University wants to test whether an online tutorial (treatment 1) is more effective than a textbook (treatment 2) at helping students learn statistics. During class, she will have each student take a pre-test, complete the online tutorial or textbook exercises, and then complete a post-test. The response of interest is the increase in test score from pre-test to post-test. Because she teaches three separate sections of the class she is considering using a randomized complete block design (RCBD), and treating each class section as a block. There are 30 students in each section of her class. Based on previous studies, she expects that $\sigma^2_{CRD} = 35$, and $\sigma^2_{RCBD} = 32$. The RCBD has the form

$$y_{ijk} = \theta_i + \beta_j + \epsilon_{ijk}, i = 1, \ldots, a, j = 1, \ldots, b, k = 1, \ldots, r$$

where $\epsilon_{ijk} \sim N(0, \sigma^2)$, and all $\epsilon' s$ are independent. Here, r is the number of replications per treatment within each block, so that the total number of observataions is $N = abr$. Note that this situation is slightly different than the one we discussed in class because we now have replicates in the RCBD. The error degrees of freedom for this design is $abr - a - b + 1$, which is equal to the degrees of freedom from SS_{AB} and SS_E in the two-factor factorial design. Note also that we are assuming that there is no interaction effect between treatment and block.

If the instructor wants a $(1 - \alpha)\%$ confidence interval around $\theta_1 - \theta_2$ that is as small as possible, should she use a CRD or a RCBD? Please justify your answer.

2 Laundry Detergent Experiment (2^2 factorial design)

A researcher for a laundry detergent company wants to look at the effect of detergent concentration and stain type on the time it takes for the stain to be removed. In a preliminary study, he sets up a 2^2 factorial experiment to look at 2 levels of detergent concentration (factor A), either 3 teaspoons or 5 teaspoons, and 2 levels of stain type (factor B), either blue ink or tomato sauce.

Results from the experiment are in the text file: http://www.public.iastate.edu/∼gdancik/stat430/detergent.txt

Each row of the table corresponds to one run of the experiment, where column A contains the coded value of factor A (with -1 being 3 teaspoons), column B contains the coded value of factor B (with -1 being blue ink), and column y contains the time until the stain is removed (in seconds).

The standard 2^2 factorial effects model is

$$y_{ijk} = \mu + \alpha_i + \beta_j + \epsilon_{ijk}, i = 1, 2, j = 1, 2, k = 1, \ldots, r$$

where $\epsilon_{ij} \sim N(0, \sigma^2)$, and $\epsilon' s$ are independent.

You will probably want to use R to help you complete the problems below.

1. Calculate the main effects of factors A and B as well as the AB interaction effect
2. Find estimates for the parameters $\mu, \alpha_i, \beta_j, (\alpha\beta)_{ij}$ for $i = 1, 2,\ j = 1, 2$. Using the fitted model, how long do you expect it would take to remove a blue ink stain using 3 teaspoons of the detergent?
3. Perform an F-test that tests the null hypothesis
   H_0: $(\alpha\beta)_{ij} = 0$ for all i, j (there is no interaction effect) versus
   H_1: $(\alpha\beta)_{ij} \neq 0$ for some i, j
   What is your conclusion, and practically, what does this result mean?

3 Balanced Incomplete Block Designs

The randomized complete block design (RCBD) requires that all treatments are observed within each block. However, it is not always possible that block sizes are large enough to accomodate all treatments. An example of this situation is when an experiment requires an oven or machine that should be treated as a block but can only hold a limited number of experimental units. In these cases, a balanced incomplete block design (BIBD) is typically used. In a BIBD, there are a treatments, each treatment is observed a total of r times, there are b blocks each of size k, each treatment appears once in a block or not at all, and each pair of treatments appear within a block λ times. In a BIBD, the total number of observations is $N = ar = bk$, and b must be $\geq a$.

The model for a BIBD has the form

$$y_{ij} = \mu + \alpha_i + \beta_j + \epsilon_{ij} = \theta_i + \beta_j + \epsilon_{ij}$$

where $\epsilon_{ij} \sim N(0, \sigma^2)$. The index i can take on values $1, \ldots, a$, and j can take on values 1, . . . , j, but not all combinations of (i, j) appear in the design.

Consider the following BIBD, with $a = 3, b = 3, k = 2$, and $\lambda = 2$.

Block 1	1 2
Block 2	1 3
Block 3	2 3

In this design, block 1, for example, contains treatments 1 and 2 (which would be assigned randomly to the experimental units within the block).

In a RCBD, we saw that for a contrast $c = (c_1, \ldots, c_a)$, $\sum_{i=1}^{a} c_i \overline{y}_{i.}$ is an unbiased estimator of $\sum_{i=1}^{a} c_i \theta_i$, where by the definition of a contrast, $\sum_{i=1}^{a} c_i = 0$. However, this result does not hold for BIBD's.

1. In the BIBD above, show that $\sum_{i=1}^{a} c_i \overline{y}_{i.}$ is a *biased* estimator of $\sum_{i=1}^{a} c_i \theta_i$. This is true for all BIBD's, but for this exercise you only need to show it for the BIBD above.

2. In a BIBD, an unbiased estimator for $\sum_{i=1}^{a} c_i\theta_i$ is

$$\frac{k}{\lambda a}\sum_{i=1}^{a} c_i Q_i,$$

where $Q_i = T_i - \frac{1}{k}\sum_{j=1}^{b} n_{ij}B_j, \quad T_i = \sum_{j=1}^{b} n_{ij}y_{ij}, \quad$ and $B_j = \sum_{i=1}^{a} n_{ij}y_{ij}$, and n_{ij} is 1 if treatment i is observed in block j, and 0 otherwise. The extra notation is necessary because not all combinations of (i, j) are present. But note that T_i is the sum of all observations on treatment i, and B_j is the sum of all observations in block j. In the BIBD above, show that $\frac{k}{\lambda a}\sum_{i=1}^{a} c_i Q_i$ is an unbiased estimator of $\sum_{i=1}^{a} c_i\theta_i$. Again, this is true for all BIBD's, but for this exercise you only need to show it for the BIBD above.