6.5 A Nonparametric Alternative: The Wilcoxon Signed-Rank Test

The Wilcoxon signed-rank test, which makes use of the sign and the magnitude of the rank of the differences between pairs of measurements, provides an alternative to the paired t test when the population distribution of the differences is nonnormal. The Wilcoxon signed-rank test requires that the population distribution of differences be symmetric about the unknown median M. Let D_0 be a specified hypothesized value of M. The test evaluates shifts in the distribution of differences to the right or left of D_0; in most cases, D_0 is 0. The computation of the signed-rank test involves the following steps:

1. Calculate the differences in the n pairs of observations.
2. Subtract D_0 from all the differences.
3. Delete all zero values. Let n be the number of nonzero values.
4. List the *absolute values* of the differences in increasing order, and assign them the ranks 1, . . . , n (or the average of the ranks for ties).

We define the following notation before describing the Wilcoxon signed-rank test:

n = the number of pairs of observations with a nonzero difference
T_+ = the sum of the positive ranks; if there are no positive ranks, $T_+ = 0$
T_- = the sum of the negative ranks; if there are no negative ranks, $T_- = 0$
T = the smaller of T_+ and T_-

μ_T

$$\mu_T = \frac{n(n+1)}{4}$$

σ_T

$$\sigma_T = \sqrt{\frac{n(n+1)(2n+1)}{24}}$$

Wilcoxon Signed-Rank Test

H_0: The distribution of differences is symmetrical around D_0. (D_0 is specified; usually D_0 is 0.)

H_a: 1. The differences tend to be larger than D_0.
2. The differences tend to be smaller than D_0.
3. Either 1 or 2 is true (two-sided H_a).

($n \le 50$)

T.S.: 1. $T = T_-$
2. $T = T_+$
3. T = smaller of T_+ and T_-

R.R.: For a specified value of α (one-tailed .05, .025, .01, or .005; two-tailed .10, .05, .02, .01) and fixed number of nonzero differences n, reject H_0 if the value of T is less than or equal to the appropriate entry in Table 6 in the Appendix.

($n > 50$)

T.S.: Compute the test statistic

$$z = \frac{T - \frac{n(n+1)}{4}}{\sqrt{\frac{n(n+1)(2n+1)}{24}}}$$

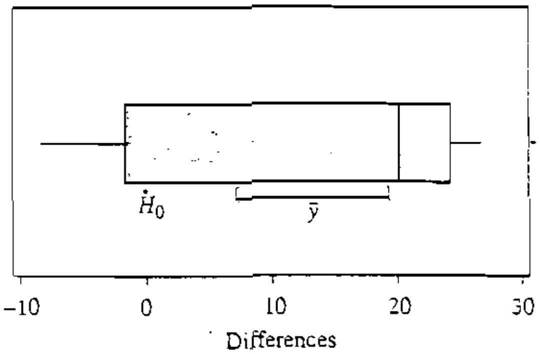
R.R.: For cases 1 and 2, reject H_0 if $z < -z_\alpha$; for case 3, reject H_0 if $z < -z_{\alpha/2}$.

Check assumptions and draw conclusions.

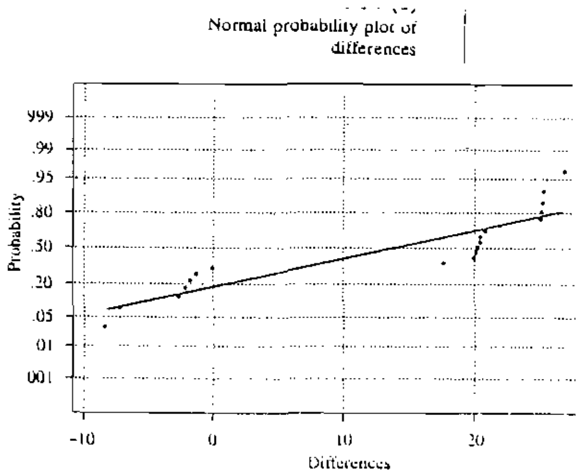
EXAMPLE 6.8

A city park department compared a new formulation of a fertilizer, brand A, to the previously used fertilizer, brand B, on each of 20 different softball fields. Each field was divided in half, with brand A randomly assigned to one half of the field and brand B to the other. Sixty pounds of fertilizers per acre were then applied to the fields. The effect of the fertilizer on the grass grown at each field was measured by the weight (in pounds) of grass clippings produced by mowing the grass at the fields over a 1-month period. Evaluate whether brand A tends to produce more grass than brand B. The data are given here.

Field	Brand A	Brand B	Difference	Field	Brand A	Brand B	Difference
1	211.4	186.3	25.1	11	208.9	183.6	25.3
2	204.4	205.7	−1.3	12	208.7	188.7	20.0
3	202.0	184.4	17.6	13	213.8	188.6	25.2
4	201.9	203.6	−1.7	14	201.6	204.2	−2.6
5	202.4	180.4	22.0	15	201.8	181.6	20.1
6	202.0	202.0	0	16	200.3	208.7	−8.4
7	202.4	181.5	20.9	17	201.8	181.5	20.3
8	207.1	186.6	20.4	18	201.5	208.7	−7.2
9	203.6	205.7	−2.1	19	212.1	186.8	25.3
10	216.0	189.1	26.9	20	203.4	182.9	20.5



Solution Evaluate whether brand A tends to produce more grass than brand B. Plots of the differences in grass yields for the 20 fields are given in Figure 6.9 (a) and (b). The differences appear to not follow a normal distribution and appear to form two distinct clusters. Thus, we will apply the Wilcoxon signed-rank test to evaluate the differences in grass yields from brand A and brand B. The null hypothesis is that the distribution of differences is symmetrical about 0 against the alternative that the differences tend to be greater than 0. First we must rank (from smallest to largest) the absolute values of the $n = 20 - 1 = 19$ nonzero differences. These ranks appear in Table 6.11.



Field	Difference	Rank of Absolute Difference	Sign of Difference	Field	Difference	Rank of Absolute Difference	Sign of Difference
1	25.1	15	Positive	11	25.3	17.5	Positive
2	−1.3	1	Negative	12	20.0	8	Positive
3	17.6	7	Positive	13	25.2	16	Positive
4	−1.7	2	Negative	14	−2.6	4	Negative
5	22.0	14	Positive	15	20.1	9	Positive
6	0	None	Positive	16	−8.4	6	Negative
7	20.9	13	Positive	17	20.3	10	Positive
8	20.4	11	Positive	18	−7.2	5	Negative
9	−2.1	3	Negative	19	25.3	17.5	Positive
10	26.9	19	Positive	20	20.5	12	Positive

The sum of the positive and negative ranks are

$$T_- = 1 + 2 + 3 + 4 + 5 + 6 = 21$$

and

$$T_+ = 7 + 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17.5 + 17.5 + 19$$
$$= 169$$

Thus, T, the smaller of T_+ and T_-, is 21. For a one-sided test with $n = 19$ and $\alpha = .05$, we see from Table 6 in the Appendix that we will reject H_0 if T is less than or equal to 53. Thus, we reject H_0 and conclude that brand A fertilizer tends to produce more grass than brand B.